that is,

$$\int_0^\infty \frac{1}{(1+y)^2}\frac{1}{1+y^n}dy = \frac{1}{2}\int_0^\infty \frac{1}{(1+z)^2}dz = \frac{1}{2}.$$

The final result is $\dfrac{1}{2}(b-a.)$

Also solved by Arkady Alt, San Jose, CA; Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Albert Stadler, Herrliberg, Switzerland, and the proposer.

- **5227:** *Proposed by José Luis Díaz-Barrero, Polytechnical University of Catalonia, Barcelona, Spain*

Compute

$$\lim_{n\to\infty}\prod_{k=1}^{n}\left(\frac{(n+1)+\sqrt{nk}}{n+\sqrt{nk}}\right).$$

Solution 1 by Kee-Wai Lau, Hong Kong, China

Since $\ln(1+x) = x + O\left(x^2\right)$ as $x \longrightarrow 0$, so

$$\sum_{k=1}^{n}\ln\left(1+\frac{1}{n+\sqrt{nk}}\right) = \sum_{k=1}^{n}\frac{1}{n+\sqrt{nk}} + O\left(\frac{1}{n}\right).$$

Hence,

$$\lim_{n\to\infty}\sum_{k=1}^{n}\ln\left(1+\frac{1}{n+\sqrt{nk}}\right) = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\frac{1}{\left(1+\sqrt{\frac{k}{n}}\right)} = \int_0^1 \frac{dx}{1+\sqrt{x}}.$$

By the substitution $x = y^2$, we easily evaluate the last integral to be $2(1-\ln 2)$.

Now by exponentiation, we find the limit of the problem to be $\dfrac{e^2}{4}$.

Solution 2 by Arkady Alt, San Jose, CA

First note that for any positive real x we have

$$e^x\left(1-\frac{x^2}{2}\right) < 1+x < e^x. \qquad (1)$$

Indeed, for any positive x we can obtain from the Taylor representation of e^x that:

$$\begin{aligned} 1+x < e^x &= 1+x+\frac{x^2}{2!}+\sum_{n=1}^{\infty}\frac{x^{n+2}}{(n+2)!} \\ &= 1+x+\frac{x^2}{2}\left(1+\sum_{n=1}^{\infty}\frac{2x^n}{(n+2)!}\right) \end{aligned}$$

$$< \quad 1 + x + \frac{x^2}{2}\left(1 + \sum_{n=1}^{\infty}\frac{x^n}{n!}\right)$$

$$= \quad 1 + x + \frac{x^2 e^x}{2} \text{ and then we have}$$

$$e^x < 1 + x + \frac{x^2 e^x}{2} \iff e^x\left(1 - \frac{x^2}{2}\right) < 1 + x.$$

Applying inequality (1) to $x = \dfrac{1}{n+\sqrt{nk}}, k = 1, 2, ..., n$ we obtain

$$e^{a_{kn}} b_{kn} < 1 + \frac{1}{n+\sqrt{nk}} < e^{a_{kn}}, k = 1, 2, ..., n, \qquad (2)$$

where $a_{kn} = \dfrac{1}{n+\sqrt{nk}}$ and $b_{kn} = 1 - \dfrac{1}{2\left(n+\sqrt{nk}\right)^2}$.

Let $S_n = \sum_{k=1}^{n} a_{kn}$. Hence,

$$e^{S_n}\prod_{k=1}^{n} b_{kn} < \prod_{k=1}^{n}\left(\frac{(n+1)+\sqrt{nk}}{n+\sqrt{nk}}\right) < e^{S_n}.$$

Note that $\lim\limits_{n\to\infty}\prod\limits_{k=1}^{n} b_{kn} = 1$. Indeed, since $n < n + \sqrt{nk} < 2n, k = 1, 2, ..., n$ then

$$1 - \frac{1}{2n^2} < 1 - \frac{1}{2\left(n+\sqrt{nk}\right)^2} < 1 - \frac{1}{8n^2}, k = 1, 2, ..., n$$

and we obtain

$$\left(1 - \frac{1}{2n^2}\right)^n < \prod_{k=1}^{n} b_{kn} < \left(1 - \frac{1}{8n^2}\right)^n < 1.$$

Since

$$\lim_{n\to\infty}\left(1 - \frac{1}{2n^2}\right)^{n^2} = \frac{1}{\sqrt{e}} \text{ then}$$

$$\lim_{n\to\infty}\left(1 - \frac{1}{2n^2}\right)^n = \lim_{n\to\infty}\sqrt[n]{\left(1 - \frac{1}{2n^2}\right)^{n^2}} = 1.$$

Since

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{1+\sqrt{\frac{k}{n}}}$$

$$= \int_0^1 \frac{1}{1+\sqrt{x}}dx = \left[x = t^2; dx = 2tdt\right]$$

$$= \quad 2\int_0^1 \frac{t}{1+t}dt = 2\left(t - \ln\left(1+t\right)\right)|_0^1 = 2\left(1-\ln 2\right), \text{ then}$$

$$\lim_{n\to\infty} e^{S_n} = \lim_{n\to\infty} e^{S_n}\prod_{k=1}^{n} b_{kn} = e^{2(1-\ln 2)} \;\; = \frac{e^2}{4}.$$

By the Squeeze Principle we see that

$$\lim_{n\to\infty}\prod_{k=1}^{n}\left(\frac{(n+1)+\sqrt{nk}}{n+\sqrt{nk}}\right) = \frac{e^2}{4}.$$

Solution 3: by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain.

The proposed limit may be written as $L = \lim_{n\to\infty}\prod_{k=1}^{n}\left(1+\frac{\frac{1}{n}}{1+\sqrt{\frac{k}{n}}}\right)$. So,

$\ln L = \lim_{n\to\infty}\sum_{k=1}^{n}\ln\left(1+\frac{\frac{1}{n}}{1+\sqrt{\frac{k}{n}}}\right)$. Now we expand each of the logs according to its power series and write this as a double sum. Then we change order of summation and sum up by columns. This is allowed because both directions provide convergent sums. So

$$\ln\left(1+\frac{\frac{1}{n}}{1+\sqrt{\frac{1}{n}}}\right) = \frac{\frac{1}{n}}{1+\sqrt{\frac{1}{n}}} - \frac{\left(\frac{\frac{1}{n}}{1+\sqrt{\frac{1}{n}}}\right)^2}{2} + \frac{\left(\frac{\frac{1}{n}}{1+\sqrt{\frac{1}{n}}}\right)^3}{3} + \cdots.$$

$$\ln\left(1+\frac{\frac{1}{n}}{1+\sqrt{\frac{2}{n}}}\right) = \frac{\frac{1}{n}}{1+\sqrt{\frac{2}{n}}} - \frac{\left(\frac{\frac{1}{n}}{1+\sqrt{\frac{2}{n}}}\right)^2}{2} + \frac{\left(\frac{\frac{1}{n}}{1+\sqrt{\frac{2}{n}}}\right)^3}{3} + \cdots.$$

$$\ln\left(1+\frac{\frac{1}{n}}{1+\sqrt{\frac{3}{n}}}\right) = \frac{\frac{1}{n}}{1+\sqrt{\frac{3}{n}}} - \frac{\left(\frac{\frac{1}{n}}{1+\sqrt{\frac{3}{n}}}\right)^2}{2} + \frac{\left(\frac{\frac{1}{n}}{1+\sqrt{\frac{3}{n}}}\right)^3}{3} + \cdots.$$

Note that

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\frac{1}{n}}{1+\sqrt{\frac{k}{n}}} = \int_0^1\frac{1}{1+\sqrt{x}}dx = \ln\left(\frac{e^2}{4}\right),$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\left(\frac{\frac{1}{n}}{1+\sqrt{\frac{k}{n}}}\right)^m}{m} = 0, \text{ for } m>1.$$

From where $\ln L = \ln\left(\frac{e^2}{4}\right)$, and therefore $L = \frac{e^2}{4}$.

Also solved by Bruno Salgueiro Fanego Viveiro, Spain; Enkel Hysnelaj, University of Technology, Sydney, Australia and Elton Bojaxhiu, Kriftel,